Symmetric Binary Forms and Involutions.

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§ 1. *Introduction.*

An involution determined by $k+1$ linearly independent binary forms of order n,

$$(\alpha_1 x)^n,\ (\alpha_2 x)^n,\ \ldots,\ (\alpha_{k+1} x)^n,$$

is an aggregate of sets of n points in the binary domain, any k of which may be chosen at random, the remaining $n-k$ being then uniquely determined. We shall say that such an involution is of *order* n, of *freedom* k, and of *extent* $n-k$; and indicate it by the symbol $I_{k,\,n-k}$.*

According to Gordan,† every combinant of the given $k+1$ forms is a comitant of the so-called "fundamental combinant"

$$F \equiv \begin{vmatrix} (\alpha_1 x_1)^n & (\alpha_2 x_1)^n & \ldots & (\alpha_{k+1} x_1)^n \\ (\alpha_1 x_2)^n & (\alpha_2 x_2)^n & \ldots & (\alpha_{k+1} x_2)^n \\ \cdots & \cdots & \cdots & \cdots \\ (\alpha_1 x_{k+1})^n & (\alpha_2 x_{k+1})^n & \ldots & (\alpha_{k+1} x_{k+1})^n \end{vmatrix},$$

a form in $k+1$ variables, $x_1, x_2, \ldots, x_{k+1}$. This theorem is fairly evident from the interpretation of F. For, on the one hand, a combinant is by definition a comitant of the involution itself, rather than of the particular forms which determine it. On the other hand, the form F, equated to zero, is an analytic statement in invariant (*i. e.*, symbolic) form of the given involution. For it is the condition that the $k+1$ points $x_1, x_2, \ldots, x_{k+1}$ belong to a set of the involution; or if the k points $x_1, x_2, \ldots, x_k$ be given, it is the equation of order n satisfied by the k given values and the $n-k$ which they determine in the involution.

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* Rather than by the usual unsymmetric symbol I^n_{n-k}.

† *Math. Annalen*, Vol. V.

Since F vanishes when x_i is x_j, where $i, j = 1, 2, \ldots, k+1$, and $i \neq j$, it contains every factor of the type $(x_i x_j)$. The other factor can be expressed symbolically as

$$H_{k, n-k} \equiv (a_1 x_1)^{n-k} (a_2 x_2)^{n-k} (a_3 x_3)^{n-k} \ldots (a_{k+1} x_{k+1})^{n-k},$$

the symbols $a_1, \ldots, a_{k+1}$ having a meaning only when all are combined to the degree $n-k$. In the identity, $F \equiv H_{k, n-k} \cdot \Pi(x_i x_j)$, F and Π each merely change in sign when x_i and x_j are interchanged. Hence $H_{k, n-k}$ is a symmetric form in the $k+1$ sets of variables.

The symmetric form $H_{k, n-k}$ thus determined by and determining an $I_{k, n-k}$ is not the most general form of its type. It will not usually happen that, when k points $x_1, x_2, \ldots, x_k$ are given and $n-k$ points determined by the equation $H_{k, n-k} = 0$ of degree $n-k$ in x_{k+1}, of the n points any set of k points will determine the remaining $n-k$ points. If, however, such a set of n points occurs for a given symmetric $H_{k, n-k}$, it will be called an *involutive set* of $H_{k, n-k}$. Evidently the extreme cases are, (*a*) the form $H_{k, n-k}$ has no involutive sets; (*b*) the form $H_{k, n-k}$ has ∞^k involutive sets. In the latter case, the involutive sets constitute an $I_{k, n-k}$, and the special form $H_{k, n-k}$ will also be denoted by $I_{k, n-k}$.

In many problems of projective geometry it is desirable to know how many involutive sets the given form $H_{k, n-k}$ possesses, and to find the conditions on the form that this number should increase until it attains its maximum, ∞^k. These questions are answered in part in this article and the results are interpreted geometrically. Two distinct geometrical representations of the form $H_{k, n-k}$ will be given first.

§ 2. *The Apolarity* Spread of* $H_{k, n-k}$.

Let

(1) $$(cx)^m = (c_1 x_2 - c_2 x_1)^m = (-1)^m [c_2^m x_1^m - \tbinom{m}{1} c_2^{m-1} c_1 x_1^{m-1} x_2 + \ldots]$$

represent symbolically the binary form of order m

(2) $$(-1)^m [c_0 x_1^m - \tbinom{m}{1} c_1 x_1^{m-1} x_2 + \tbinom{m}{2} c_2 x_1^{m-2} x_2^2 - \tbinom{m}{3} c_3 x_1^{m-3} x_2^3 \ldots].$$

* A conspicuous example of the representation here employed is contained in the memoirs of Study, *Math. Annalen*, Vols. XXVII and XL. There the ternary quadratic is taken as the element, rather than the binary m-ic as above. Much of the content of this paragraph is implied in Study's Ternäre Formen, where (p. x) also there is found the notation for binary forms.

By the equations

$$(3) \qquad X_0 = c_0,\ X_1 = -\binom{m}{1}c_1,\ \ldots,\ X_i = (-1)^i\binom{m}{i}c_{m-i}, \ldots$$

the form is represented by a point X in S_m and conversely. All m-ics, $(cx)^m$, apolar to a given m-ic, $(dx)^m$, satisfy the linear condition, $(cd)^m = 0$, or, from (3),

$$(4) \qquad X_0 d_m + X_1 d_{m-1} + X_2 d_{m-2} + \ldots + X_m d_0 = 0;$$

i. e., the m-ics, $(cx)^m$, are represented by points X on an S_{m-1}, U, in S_m, whose coordinates are

$$(5) \qquad U_0 = d_m,\ U_1 = d_{m-1}, \ldots,\ U_m = d_0.$$

Thus the m-ic, $(dx)^m$, can also be represented by an S_{m-1}, U, and conversely. Wherever it is necessary to avoid confusion, we shall suppose that U represents $(cx)^m$ and X represents $(c\bar{x})^m$. The incidence condition for point X and space U is the apolarity condition for the corresponding binary forms.

To perfect m-th powers, $(y\bar{x})^m$, correspond points on the fundamental norm-curve, N_m, whose parametric equation in point form is

$$(6) \qquad X_0 = y_2^m,\ X_1 = -\binom{m}{1}y_2^{m-1}y_1,\ X_2 = \binom{m}{2}y_2^{m-2}y_1^2, \ldots.$$

The parametric equation of the norm-curve in reciprocal form, N_m, is

$$(7) \qquad U_0 = y_1^m,\ U_1 = y_1^{m-1}y_2,\ U_2 = y_1^{m-2}y_2^2, \ldots.$$

The space U of N_m osculates N_m at a point X with the same parameter. The equation $(cx)^m = 0$ determines the parameters of the m points X where the space $(cx)^m$ cuts N_m and also the parameters of the m spaces U of N_m which pass through the point $(c\bar{x})^m$.

Just as the form $(cx)^m$ is represented by an S_{m-1}, U, the locus of points X which represent forms $(d\bar{x})^m$ apolar to $(cx)^m$, so the symmetric form

$$(8) \qquad H_{1,m} \equiv (a_1x_1)^m(a_2x_2)^m = (a_1x_2)^m(a_2x_1)^m$$

is represented by a point-quadric. The points $(c\bar{x})^m$ and $(d\bar{x})^m$ are an apolar point-pair of the quadric if $(a_1c)^m(a_2d)^m = 0$; and the point $(c\bar{x})^m \equiv (c'\bar{x})^m$ is on the quadric if $(a_1c)^m(a_2c')^m = 0$. In particular, if x_1 and x_2 are values that satisfy $H_{1,m} = 0$, they are the parameters of two points on N_m, apolar to the quadric. The equation $(a_1x)^m(a_2x)^m = 0$ gives the parameters of the $2m$ points in which the quadric cuts N_m.

The same symmetric form can be considered dually to represent an S_{m-1}-quadric and will then be written with dashed variables. Evidently the

point-quadric $(a_1x_1)^m(a_2x_2)^m$ and the S_{m-1}-quadric $(\alpha_1\bar{x}_1)^m(\alpha_2\bar{x}_2)^m$ are apolar if $(a_1\alpha_1)^m(a_2\alpha_2)^m=0$.

The quadrics determined by the special symmetric forms

$$Q_i=(x_1x_2)^{2i}(a_1x_1)^{m-2i}(a_2x_2)^{m-2i} \qquad 0<2i\leqq m$$

are important. To characterize them we observe that the condition

$$(cc')^{2i}(a_1c)^{m-2i}(a_2c')^{m-2i}=0$$

is satisfied when

$$(cx)^m=(c'x)^m=(s_1x)\,.\,(s_2x)\,.\,\ldots\,.\,(s_{i-1}x)\,.\,(tx)^{m-i+1}$$

for all values of $s_1, s_2, \ldots, s_{i-1}$, and t; and that when

$$(cx)^m=(c'x)^m=(s_1x)\,.\,(s_2x)\,.\,\ldots\,.\,(s_ix)\,.\,(tx)^{m-i},$$

the condition reduces to

$$(s_1t)^2\,.\,(s_2t)^2\,.\,\ldots\,.\,(s_it)^2\,.\,(a_1t)^{m-2i}(a_2t)^{m-2i}=0.$$

(9) *The point-quadric Q_i contains N_m and all its osculating spaces up to and including those of dimensions $i-1$; it also contains the $2(m-2i)$ osculating spaces of dimension i whose parameters t are determined by $(a_1t)^{m-2i}(a_2t)^{m-2i}=0$.*

A dual statement holds for the S_{m-1}-quadric whose form is

$$Q_i=(\bar{x}_1\bar{x}_2)^{2i}(\alpha_1x_1)^{m-2i}(\alpha_2x_2)^{m-2i}.$$

To interpret the Gordan expansion of the symmetric form, we write in the usual way

$$(10)\quad (a_1x_1)^m(a_2x_2)^m=\sum_{i=0}^{i=\frac{m}{2};\frac{m}{2}-1}\frac{\binom{m}{2i}\binom{m}{2i}}{\binom{2m-2i+1}{2i}}(x_1x_2)^{2i}\left\{(a_1a_2)^{2i}(a_1x_1)^{m-2i}(a_2x_1)^{m-2i}\right\}_{\substack{m-2i\\x_2}}$$

where the subscript of the brace indicates the $(m-2i)$-th polar of the enclosed form as to x_2. Denoting the various terms of the expansion and the corresponding quadrics by Π_i, we have

$$(11)\qquad (a_1x_1)^m(a_2x_2)^m=\Pi_0+\Pi_1+\Pi_2+\ldots+\Pi_{\frac{m}{2};\frac{m-1}{2}}.$$

Evidently the form of the last term varies according as m is even or odd.

The same form (10) with the dual meaning is expanded in the same way and we have

$$(12)\qquad (a_1\bar{x}_1)^m(a_2\bar{x}_2)^m=P_0+P_1+P_2+\ldots+P_{\frac{m}{2};\frac{m-1}{2}}.$$

The quadric Π_i is of the type Q_i described in (9) but is further specialized by the fact that its form is a product of $(xy)^{2i}$ and a *polarized* form. This further condition is equivalent to the apolarity of the form with any form containing the factor $(xy)^{2i+2k}$, where $k \geqq 1$. Thus a geometric statement of the expansion is:

(13) *With regard to a given norm-curve N_m, any quadric can be expressed uniquely as a sum of* $\frac{m}{2}+1$; $\frac{m-1}{2}+1$ *quadrics Π_i each of which is of the type described in* (9) *and furthermore is apolar to every quadric of the type P_{i+k} where $k > 0$.*

The special case of this to be employed is

(14) *A symmetric form $(a_1x_1)^m(a_2x_2)^m$ is a polarized* $2m$-ic *when and only when the corresponding quadric is apolar to all the quadrics inscribed to the norm-curve.*

To express certain conditions it is necessary to have the reciprocal equation or form of the point quadric given by the form $(a_1x_1)^m(a_2x_2)^m$. As usual, we proceed as follows: The polar of a point $(c\bar{x})^m$ which lies on a space $(dx)^m$ as to the quadric is a space $(ex)^m$; that is,

$$\begin{aligned}(a_1c)^m(a_2x_2)^m &\equiv \rho(ex_2)^m,\\ (dc)^m &= 0.\end{aligned}$$

Eliminating the $m+2$ quantities $c_0, \ldots, c_m, \rho$ from the $m+2$ equations given by the identity and the equation, there results the reciprocal quadric polarized as to the spaces $(dx)^m$ and $(ex)^m$. The result expressed symbolically in terms of the coefficients of $(a_1x_1)^m(a_2x_2)^m = (a_1'x_1)^m(a_2'x_2)^m = \ldots$ is

(15) $$(\alpha_1\bar{x}_1)^m(\alpha_2\bar{x}_2)^m \equiv (a_1\bar{x}_1)(a_1'\bar{x}_1)\ldots(a_1^{(m-2)}\bar{x}_1)(a_2\bar{x}_2)(a_2'\bar{x}_2)\ldots (a_2^{(m-2)}\bar{x}_2)\,\Pi(a_1^{(i)}a_1^{(k)})\,\Pi(a_2^{(i)}a_2^{(k)}),$$

where $i, k = 0, 1, \ldots, m-2$ and $i \neq k$.

The discriminant of the given quadric can be found at once as the apolarity condition of $(a_1x_1)^m(a_2x_2)^m$ and $(\alpha_1\bar{x}_1)^m(\alpha_2\bar{x}_2)^m$. To within a numerical factor its value is

(16) $$D = \Pi(a_1^{(i)}a_1^{(k)})\,\Pi(a_2^{(i)}a_2^{(k)}) \qquad i, k = 0, 1, \ldots, (m-1) \text{ and } i \neq k.$$

Let the symmetric form be the polar of a $2m$-ic, $(hx)^{2m}$. Since $(hx)^{2m}$ can be expressed in ∞^1 ways as a sum of $(m+1)$ $2m$-th powers, say

$$(hx)^{2m} = \lambda_0(r_0x)^{2m} + \lambda_1(r_1x)^{2m} + \ldots + \lambda_m(r_mx)^{2m},$$

the symmetric form can also be expressed in ∞^1 ways as

(17) $$(hx_1)^m(hx_2)^m = \lambda_0(r_0x_1)^m(r_0x_2)^m + \lambda_1(r_1x_1)^m(r_1x_2)^m + \ldots\ldots$$

Hence the quadric is in the form $\sum_{i=0}^{i=m}\lambda_i(R_iX_1)(R_iX_2)$, where the space R_i osculates N_m at the point whose parameter is r_i. Thus the spaces R_i form a self-polar $(m+1)$-edron of the quadric, and with reference to (14) we can state that

(18) *If a proper point quadric is apolar to all the quadrics inscribed to N_m, it has ∞^1 self-polar $(m+1)$-edra whose spaces osculate N_m.*

As shown above in detail for the forms $H_{0,n-k}$ and $H_{1,n-k}$, the general symmetric form $H_{k,n-k}$ can be viewed as a spread in S_{n-k} of order $k+1$ and dimensions $n-k-1$. The general spread can be represented in this way by a symmetric form. The spread is supposed taken in points or reciprocally according as the variables are not or are dashed.

We shall define an *antorthic* r-point of a point spread of order t* to be a set of r points such that any s points of the set are apolar to the spread of order t. such a set will be indicated by the symbol $A^r_{s,t}$. Of course $s \lesseqgtr r$ and $s \lesseqgtr t$. If the $H_{k,n-k}$ has an involutive set as defined earlier, this set can be interpreted as an $A^n_{k+1,k+1}$ of the spread determined by the form lying on the norm-curve N_{n-k}. One of the problems to be considered concerns the distribution of these $A^n_{k+1,k+1}$ upon norm-curves.

§ 3. *The Parametric† Spread of $H_{k,n-k}$.*

In $H_{k,n-k}$, let $x_1, x_2, \ldots, x_{k+1}$ be $k+1$ points on a norm-curve in a space S_{k+1}. The set of $k+1$ values x_i determine a point X in S_{k+1}. If the set be chosen so that $H_{k,n-k}=0$, the point X lies on a spread in S_{k+1} of order $n-k$ and dimensions k. Again the dual interpretation is indicated by dashed variables. The general spread in S_{k+1} can be represented in this way. For if in the equation of the spread the coordinates of the variable point X be expressed in terms of the symmetric combinations of the parameters $x_1, x_2, \ldots, x_{k+1}$, the equation takes the form $H_{k,n-k}=0$.

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* An *orthic set* of r linear forms (German, "Pol-r-Seite") has been defined as a set in terms of whose powers a given form can be expressed. The antorthic r-point (German "Pol-r-Eck") may determine an orthic set by taking all the linear forms incident with the r points.

† The representation here given is the one usually employed: Meyer's *Apolarität* or Grace and Young.

An *inscribed n-space* of the spread is a set of n linear spaces S_k in S_{k+1} any $k+1$ of which meet in a point of the spread. If $H_{k,n-k}$ has an involutive set, the set represents an n-space inscribed in the spread and circumscribed to the norm-curve, N_{k+1}.

The spread in question will be denoted by Ψ_k^{n-k} of the form $H_{k,n-k}$; the spread of § 2, by Φ_{n-k-1}^{k+1} of $H_{k,n-k}$. Since each is general any spread can be represented in either way. Thus the Ψ_k^{n-k} of $H_{k,n-k}$ is the Φ_k^{n-k} of a symmetric form $H_{n-k-1,k+1}$. This form is obtained from $H_{k,n-k}$ by replacing the symmetric combinations of $x_1, x_2, \ldots, x_{k+1}$ by the coefficients of a binary form $(cx)^{k+1}=(c'x)^{k+1}=\ldots$; in the result of degree $n-k$ in the coefficients of $(cx)^{k+1}$, the coefficients of $n-k$ forms $(c_1x)^{k+1}, \ldots, (c_{n-k}x)^{k+1}$ are introduced by the Aronhold process; and finally the cofficients of the $n-k$ forms are replaced by $n-k$ sets of variables $x_1, x_2, \ldots, x_{n-k}$. By the inverse process the Φ_{n-k-1}^{k+1} of $H_{k,n-k}$ can be expressed as the Ψ_{n-k-1}^{k+1} of a symmetric form $H_{n-k-1,k+1}$. Only when $n=2k+1$ do the spreads Φ and Ψ of the *same* form $H_{k,n-k}$ have the same order and dimension. Though still distinct, they are covariants of each other and the norm-curve.

§ 4. *The Symmetric Form,* $H_{1,n-1}$.

The Φ_{n-2}^2 of $H_{1,n-1}=(a_1x_1)^{n-1}(a_2x_2)^{n-1}$ is a quadric in S_{n-1} considered with respect to a norm-curve N_{n-1}. If $H_{1,n-1}$ has an involutive set of n points, there is on N_{n-1} an antorthic set $A_{2,2}^n$ of Φ_{n-2}^2. Hence Φ in reciprocal form is apolar to all the point quadrics containing N_{n-1}. According to the dual of (18) there are then $\infty^1 A_{2,2}^n$ of Φ on N_{n-1}. Hence $H_{1,n-1}$ has ∞^1 involutive sets; *i. e.*,

(19) *If the symmetric form $H_{1,n-1}$ has a single involutive set of n points, it has ∞^1 sets and is an $I_{1,n-1}$.*

The Ψ_1^{n-1} of $H_{1,n-1}$ is a curve of order $n-1$ in S_2 with reference to a conic N_2. If $H_{1,n-1}$ has an involutive set, there is an n-line inscribed in Ψ and circumscribed about N_2. The translation of (19) is:

(20) *If an n-line circumscribed about a conic is inscribed in a curve of order $n-1$, there are ∞^1 n-lines circumscribed about the conic and inscribed in the curve whose n points of contact form an $I_{1,n-1}$ on the conic.*

The theorem of Stroh* that, after multiplication by an invariant, all the

* *Math. Annalen*, Vol. XXXIV, p. 323.

combinants of two binary forms of order n can be expressed as covariants of a single binary form of order $2(n-1)$ is also evident. For the condition that $H_{1,n-1}$ represent an involution is that the quadric Φ^2_{n-2} *in reciprocal form* be apolar to all the point-quadrics containing N_{n-1}. According to (14) this reciprocal form is then merely a polarized form of order $2(n-1)$, say $(ax_1)^{n-1}(ax_2)^{n-1}$. But from the reciprocal form, the original form $H_{1,n-1}$ can be recovered, multiplied however by the $(n-2)$th power of the discriminant of Φ which is the resultant of the two given binary forms. Thus the fundamental combinant, $H_{1,n-1}$, is itself a covariant of $(ax)^{2(n-1)}$. The analytical results of Stroh follow readily from formulae (15) and (16).

§ 5. *Theorems Concerning Involutive Sets.*

(21) *The symmetric form $H_{k,n-k}$ is an $I_{k,n-k}$ if, for arbitrarily assigned values of $k-1$ variables $x_1, x_2, \ldots, x_{k-1}$, the form $H_{1,n-k}$ in the other two variables, x_k and x_{k-1}, is an $I_{1,n-k}$.*

Let the values $y_1, \ldots, y_k$ of k of the variables determine the values $z_{k+1}, z_{k+2}, \ldots, z_n$ of the remaining variable; *i. e.*, the $k+1$ values $y_1, \ldots, y_k$, z_{k+l}, where $l=1, 2, \ldots, n-k$, satisfy $H_{k,n-k}=0$. Since, for the given values $y_1, y_2, \ldots, y_{k-1}$, the form is an $I_{1,n-k}$, then also $y_1, y_2, \ldots, y_{k-1}, z_{k+l}, z_{k+m}$ satisfy $H_{k,n-k}=0$. Hence the values $y_1, \ldots, y_{k-1}, z_{k+l}$ determine the values y_k and z_{k+m} where $m \neq l$. Again, for given values $y_1, \ldots, y_{k-2}, z_{k+l}$, the form is an $I_{1,n-k}$; hence the $(k-2)$ y's and any three z's satisfy $H_{k,n-k}=0$. By a repetition of this process one can show that any $k+1$ values selected from the y's and z's satisfy $H_{k,n-k}=0$; whence the y's and z's constitute an involutive set and all sets determined by the form are involutive.

Writing, as before,

$$H_{k,n-k} = (a_1x_1)^{n-k}(a_2x_2)^{n-k}\ldots(a_{k+1}x_{k+1})^{n-k},$$

we have already determined the condition that the form

$$H_{1,n-k} \equiv (a_kx_k)^{n-k}(a_{k+1}x_{k+1})^{n-k}$$

be an $I_{1,n-k}$; namely, if $(\alpha_k\bar{x}_k)^{n-k}(\alpha_{k+1}\bar{x}_{k+1})^{n-k}$ of degree $n-k$ in the coefficients of $H_{1,n-k}$ be the quadric reciprocal to $H_{1,n-k}$ (in the sense of § 2), then the identical vanishing of $(\alpha_k\alpha_{k+1})^2(\alpha_k\bar{x}_k)^{n-k-2}(\alpha_{k+1}\bar{x}_{k+1})^{n-k-2}$ is the required con-

dition. Expressing that this condition is satisfied for all values of $x_1, x_2, \ldots, x_{k-1}$, we have

$$(22) \quad (a_1x_1)^{n-k}(a_1'x_1)^{n-k}\ldots(a_1^{(n-k-1)}x_1)^{n-k}\ldots(a_{k-1}x_{k-1})^{n-k}(a_{k-1}'x_{k-1})^{n-k} \ldots (a_{k-1}^{(n-k-1)}x_{k-1})^{n-k}\,.\,(\alpha_k\alpha_{k+1})^2(\alpha_k\bar{x}_k)^{n-k-2}(\alpha_{k+1}\bar{x}_{k+1})^{n-k-2}\equiv 0.$$

Hence

(23) *The identical vanishing of the form* (22) *of degree* $n-k$ *in the coefficients of* $H_{k,n-k}$, *symmetrical of order* $(n-k)^2$ *in* $k-1$ *variables, and symmetrical of order* $n-k-2$ *in two variables, is the necessary and sufficient condition that* $H_{k,n-k}$ *be an* $I_{k,n-k}$.

Naturally this condition does not apply to the case $n-k=1$. For then mere symmetry is sufficient to require an $I_{k,1}$ and the form $H_{k,1}$ is a completely polarized $(k+1)$-ic. The most convenient application is to the case $n-k=2$. Then the two variables $\bar{x}_k$ and $\bar{x}_{k+1}$ disappear and the condition is a symmetric form, $H_{k-2,4}=0$. Most of what follows is concerned with this case.

Specializing the results just obtained we have

(24) *If* $H_{1,2}=(a_1x_1)^2(a_2x_2)^2=(a_1'x_1)^2(a_2'x_2)^2$ *has one involutive set, it has* ∞^1 *sets and is an* $I_{1,2}$. *The invariant condition for this is*

$$(a_1a_1')(a_2a_2')\{(a_1a_2)(a_1'a_2')+(a_1a_2')(a_1'a_2)\}=0.$$

From this theorem we have further:

(25) *The form* $H_{2,2}\equiv(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2=(a_1'x_1)^2(a_2'x_2)^2(a_3'x_3)^2=\ldots$ *has in general a single involutive set given by the quartic*

$$(a_1a_1')(a_2a_2')\{(a_1a_2)(a_1'a_2')+(a_1a_2')(a_1'a_2)\}(a_3x)^2(a_3'x)^2=0.$$

If the form has more than one involutive set, it has ∞^2 *sets and is an* $I_{2,2}$. *The invariant condition for this is the identical vanishing of the quartic.*

For if y_1, y_2, y_3, y_4 are the roots of the quartic, they are the values for which $H_{2,2}$ becomes an $I_{1,2}$ (24). Let $H_{2,2}$, when $x_1=y_1$ and $x_2=y_2$, determine the values z_3 and z_4 of x_3. Since, when $x_1=y_1$, the form is an $I_{1,2}$ and is satisfied by y_1, y_2, z_3 and y_1, y_2, z_4, it is also satisfied by y_1, z_3, z_4. Similarly it is satisfied by y_2, z_3, z_4. Hence y_1, y_2, z_3, z_4 are an involutive set of $H_{2,2}$ and for any one of the four, $H_{2,2}$ reduces to an $I_{1,2}$. That is, z_3 and z_4 are y_3 and y_4. If $H_{2,2}$ has more than one involutive set, the quartic must vanish identically. But according to (23), $H_{2,2}$ is then an $I_{2,2}$.

Again, let

$$H_{3,2}\equiv(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2(a_4x_4)^2=(a_1'x_1)^2(a_2'x_2)^2(a_3'x_3)^2(a_4'x_4)^2=\ldots\ldots$$

If, for assigned values of two of the variables x_1 and x_2, $H_{3,2}$ reduces to an $I_{1,2}$, x_1 and x_2 satisfy the symmetric relation

$$\overline{H}_{1,4} = (a_3a_3')(a_4a_4')\{(a_3a_4)(a_3'a_4') + (a_3a_4')(a_3'a_4)\}(a_1x_1)^2(a_1'x_1)^2(a_2x_2)^2(a_2'x_2)^2 = 0.$$

We note first that every involutive set of $H_{3,2}$ is an involutive set of $\overline{H}_{1,4}$. Conversely, every involutive set of $\overline{H}_{1,4}$ is an involutive set of $H_{3,2}$. For if $x_1, \ldots, x_5$ is an involutive set of $\overline{H}_{1,4}$, and if the value x_1 be assigned in $H_{3,2}$, reducing it to an $H_{2,2}$, the single involutive set of this $H_{2,2}$ is $x_2, \ldots, x_5$ (25). Hence x_1 and any three of $x_2, \ldots, x_5$ satisfy $H_{3,2} = 0$. Similarly with regard to x_2, etc., and the set is involutive. If $\overline{H}_{1,4}$ has one involutive set, it has ∞^1, and we conclude:

(26) *$H_{3,2}$ has in general no involutive set. If it has one involutive set, it has ∞^1 which form an $I_{1,4}$. The invariant condition for this is the condition that $\overline{H}_{1,4}$ be an $I_{1,4}$ [see (22) and (23)]. If $H_{3,2}$ has an involutive set not contained in the $I_{1,4}$, it has ∞^3 sets and is an $I_{3,2}$, the invariant condition for which is $\overline{H}_{1,4} \equiv 0$.*

From induction there follows:

(27) *The form $H_{k,2}$ has the same involutive sets as the form $\overline{H}_{k-2,4}$ whose identical vanishing is the condition that $H_{k,2}$ be an $I_{k,2}$.*

Thus a study of the forms $H_{k,2}$ when $k > 2$ involves that of the symmetric forms of order 4 in more than two variables.

§ 6. *The Form $H_{1,2}$ as a Ternary Quadratic.*

The binary quadratic $(a\bar{x})^2 = a_2^2\bar{x}_1^2 - 2a_2a_1\bar{x}_1\bar{x}_2 + a_1^2\bar{x}_2^2$ represents a point X, and the quadratic $(ax)^2$ a line U, in a plane when we put

$$(28) \qquad \begin{aligned} X_0 &= a_2^2, & X_1 &= -2a_1a_2, & X_2 &= a_1^2 \\ U_0 &= a_1^2, & U_1 &= a_1a_2, & U_2 &= a_2^2. \end{aligned}$$

The condition that three points lie on a line or that three lines pass through a point is that the corresponding quadratics be in involution; the incidence condition is the apolarity condition. Employing the usual symbolic notations for the binary and ternary domains, the formulae are

$$(29) \qquad \begin{aligned} (XYZ) &= 2(ab)(ac)(bc) \\ (UVW) &= (ab)(ac)(bc) \\ U_X &= (ab)^2. \end{aligned}$$

Ternary forms of degree higher than the first are always polarized; thus the norm-conic is

$$(30)\quad \begin{aligned} N_X N_Y &= (2X_2X_0 - \tfrac{1}{2}X_1^2)_Y = X_2Y_0 + X_0Y_2 - \tfrac{1}{2}X_1Y_1 = (x_1x_2)^2 \\ U_N V_N &= (2U_2U_0 - 2U_1^2)_V = U_2V_0 + U_0V_2 - 2U_1V_1 = (x_1x_2)^2. \end{aligned}$$

Small letters can be used for both domains since the difference in symbols distinguishes them.

The general conic, its line form, and its discriminant are at once written from (29):

$$(31)\quad \begin{aligned} a_xa_y &= (a_1x_1)^2(a_2x_2)^2 = (a_1x_2)^2(a_2x_1)^2 = (a_1'x_1)^2(a_2'x_2)^2 = \ldots, \\ (aa'u)(aa'v) &= (a_1a_1')(a_2a_2')(a_1\bar{x}_1)(a_1'\bar{x}_1)(a_2\bar{x}_2)(a_2'\bar{x}_2), \\ (aa'a'')^2 &= (a_1a_1')(a_1a_1'')(a_1'a_1'')(a_2a_2')(a_2a_2'')(a_2'a_2''). \end{aligned}$$

Thus $H_{1,2}$ appears as a conic whose apolarity relations with the norm-conic N are

$$(32)\quad \begin{aligned} a_N^2 &= (a_1a_2)^2 \\ (aa'N)^2 &= \tfrac{1}{2}(a_1a_1')(a_2a_2')\{(a_1a_2)(a_1'a_2') + (a_1a_2')(a_1'a_2)\}. \end{aligned}$$

If $H_{1,2}$ has one involutive sets, it has ∞^1 sets. Then $(aa'N)^2 = 0$ and $H_{1,2}$ is an $I_{1,2}$. The triads of $I_{1,2}$ are antorthic sets $A_{2,2}^3$ (*i. e.*, self-polar triangles) of the conic $H_{1,2}$ which lie on N.

If, for the $I_{1,2}$, $a_N^2 = 0$, the conics N and $H_{1,2}$ meet in a set of four points equianharmonic on either conic. The $I_{1,2}$ is then the second polar system of this quartic.

If, for the $I_{1,2}$, $(aa'a'')^2 = 0$, $H_{1,2}$ has a double point on N with parameter t, say. Then the form $H_{1,2}$ is $(x_1 - t)(x_2 - t) \cdot (\alpha_1x_1)(\alpha_2x_2)$, where $(\alpha_1x_1)(\alpha_2x_2)$ is a form $H_{1,1}$ such that $(\alpha_1x)(\alpha_2x) = 0$ fixes the two other meets of $H_{1,2}$ and N. In this case the $I_{1,2}$ degenerates into the neutral point t and an $I_{1,1}$.

If, for the $I_{1,2}$, $(aa'u)(aa'v)$ vanishes identically, the conic $H_{1,2}$ is the square of a line which meets N in points whose parameters are s and t. The form $H_{1,2}$ is then $(x_1 - s)(x_1 - t)(x_2 - s)(x_2 - t)$, and the $I_{1,2}$ degenerates into an arbitrary point and the neutral points s and t.

§ 7. *The Form $H_{2,2}$ as a Ternary Cubic.*

The spread of $H_{2,2}$ as defined in § 2 is a plane cubic, $a_x^3 = a_x'^3 = a_x''^3 = \ldots$, and we write

$$(33)\quad f = a_xa_ya_z = (a_1x_1)^2(a_2x_2)^2(a_3x_3)^2.$$

Involutive sets of $H_{2,2}$ are sets of 4 points on the norm-conic N such that any 3 of the 4 are apolar to the cubic, *i. e.*, antorthic sets $A^4_{3,3}$.* Using the first property of (25), we find that

(34) *On any conic there is, in general, a single antorthic 4-point of a given cubic.*

The spread of $H_{2,2}$ as defined in § 3 is a space quadric, and the translated theorem is:

(35) *There is, in general, a single tetrahedron circumscribed about a cubic space curve and inscribed in a quadric.*†

If the point x (or x_1 on N) belongs to the antorthic set, the polar conic of x as to f, when taken in lines, is apolar to N, or, symbolically, $(aa'N)^2 a_x a'_x = 0$. The meets of this conic and N^2_x form the unique antorthic set.

If the $A^4_{3,3}$ is not unique, the $H_{2,2}$ is an $I_{2,2}$, and N and f are subject to the relation

$$(aa'N)^2 a_x a'_x = \lambda N^2_x, \tag{36}$$

which expresses, in the ternary domain, that the polar conic as to f of any point x on N, when taken in line form, is apolar to N in point form; and, in the binary domain, that for any value of x, the form $H_{2,2}$ reduces to an $I_{1,2}$ and is therefore an $I_{2,2}$.

The conics N^2_x which satisfy the relation (36) have been investigated in companion papers by Professors White and Gordan.‡ The former finds, corresponding to the values $\lambda = \pm\sqrt{\tfrac{1}{6}S}$, two nets of conics, the polar conics of two irrational covariants of f,

$$A \equiv A^3_x = \Delta + \sqrt{\tfrac{1}{6}S}f, \quad \text{and} \quad B = B^3_x = \Delta - \sqrt{\tfrac{1}{6}S}f. \tag{37}$$

These conics on each of which lie ∞^2 antorthic sets of f which constitute an $I_{2,2}$ will be called *involution conics.*

An $A^4_{3,3}$ is a set of four points subject to four relations; *i. e.*, there are ∞^4 $A^4_{3,3}$. In general, an $A^4_{3,3}$ is determined uniquely when two of its points, say x and y, are given. For the other two points, z and t must lie on the polar line of

* These sets have been discussed by Caporali, *Memorie*, p. 51, § 5.

† Theorem of Meyer, *Apolarität*, p. 279 (y_1). The parametric representation is treated there quite fully, and further results in this direction will not be given.

‡ *Trans. American Math. Soc.*, Vol. I, pp. 1 and 9. Other references are given there. The notation for the comitants of f is that of Gordan.

xy as to f and must be harmonic with the pairs of points in which this polar line meets the polar conics of x and y. Through x and y there passes a definite polar conic of A and of B. Since each is an antorthic conic, the polar line of xy as to f must meet each in a pair of points which form with x and y an $A^4_{3,3}$ of f. Hence these conics meet in the $A^4_{3,3}$, x, y, z, t.

(38) *The ∞^4 $A^4_{3,3}$'s of a cubic f are distributed ∞^2 at a time on the ∞^2 polar conics of A; and in the same way on the ∞^2 polar conics of B. On a fixed polar conic of the one cubic, the $A^4_{3,3}$'s lie in an $I_{2,2}$ which is cut out by all the polar conics of the other cubic.*

Assuming henceforth that $H_{2,2}$ is an $I_{2,2}$ on an involution conic N^2_x we have the following parallel between the comitants (polarized) of f and the comitants of the form $I_{2,2}$, the transition being made at once by the use of (29):

$$
\begin{aligned}
(39)\quad N &= N_x N_y &&= (x_1x_2)^2,\\
f &= a_x a_y a_z &&= (a_1x_1)^2(a_2x_2)^2(a_3x_3)^2,\\
\theta &= (aa'u)(aa'v)a_x a'_y &&= (a_1x_1)^2(a'_1x_2)^2(a_2a'_2)(a_3a'_3)(a_2\bar{x}_1)(a'_2\bar{x}_1)(a_3\bar{x}_2)(a'_3\bar{x}_2),\\
\Delta &= (aa'a'')^2 a_x a'_y a''_z &&= (a_1x_1)^2(a'_1x_2)^2(a''_1x_3)^2(a_2a'_2)(a_2a''_2)\\
&&&\qquad (a'_2a''_2)(a_3a'_3)(a_3a''_3)(a'_3a''_3)\\
&&&= (\delta_1x_1)^2(\delta_2x_2)^2(\delta_3x_3)^2.
\end{aligned}
$$

Then the relation (36) becomes, in the binary domain,

$$(40)\quad \tfrac{1}{2}(a_1x_1)^2(a'_1x_2)^2(a_2a'_2)(a_3a'_3)[(a_2a_3)(a'_2a'_3)+(a_2a'_3)(a'_2a_3)] = \sqrt{\tfrac{1}{6}S}(x_1x_2)^2.$$

Expanding the left-hand member in powers of (x_1x_2), the first term is a polar of the binary quartic in (25) which is now identically zero. The second term is an invariant multiplied by $(x_1x_2)^2$, and equating coefficients we have

$$(41)\quad +\sqrt{\tfrac{1}{6}S} = \tfrac{1}{6}(a_1a')^2(a_2a'_2)(a_3a'_3)[(a_2a_3)(a'_2a'_3)+(a_2a'_3)(a'_2a_3)].$$

Thus *the irrational invariant $\sqrt{\frac{1}{6}S}$ of f is a rational invariant of the form $I_{2,2}$.* This is to be expected, since the irrationality is really adjoined by the choice of N^2_x as an involution conic. Noting that a polar conic of $A = \Delta + \sqrt{\frac{1}{6}S}f$ is one which satisfies the relation (36) for the value $\lambda = +\sqrt{\frac{1}{6}S}$, since

$$A_x(aa'A)^2a_ya'_y = +\sqrt{\tfrac{1}{6}S}[\Delta_x\Delta^2_y + \sqrt{\tfrac{1}{6}S}f_xf^2_y] = +\sqrt{\tfrac{1}{6}S}A_xA^2_y,\text{*}$$

we can write

$$
\begin{aligned}
(42)\quad A &= \Delta + \sqrt{\tfrac{1}{6}S}f = (\delta_1x_1)^2(\delta_2x)^2(\delta_3x_3)^2 + \sqrt{\tfrac{1}{6}S}(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2,\\
B &= \Delta - \sqrt{\tfrac{1}{6}S}f = (\delta_1x_1)^2(\delta_2x)^2(\delta_3x_3)^2 - \sqrt{\tfrac{1}{6}S}(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2\\
&= (b_1x_1)^2(b_2x_2)^2(b_3x_3)^2,
\end{aligned}
$$

* The symbolic calculations not explicitly given follow readily from the formulae of Gordan (*loc. cit.*) or from those in Clebsch-Lindemann, Vol. II (French edition).

where of course $\sqrt{\frac{1}{6}S}$, when it occurs in a binary expression, is understood to be the binary invariant of (41). *The choice of the positive sign in* (41) *requires that* A *in* (42) *be the cubic whose polar conic is* $N = (x_1x_2)^2$.

When the $I_{2,2}$ on N is given, the cubic f, the value $+\sqrt{\frac{1}{6}S}$, and the point P whose polar as to A is N are uniquely determined. Conversely, when f, the value $+\sqrt{\frac{1}{6}S}$, and the point P are given, the antorthic conic N and the $I_{2,2}$ on it are uniquely determined. Since the system of invariants of f and P is the system of covariants of f, we conclude that

(43) *The rational invariant theory of the involution form* $I_{2,2}$ *coincides with the rational covariant theory of the ternary cubic* f *after the adjunction of* $\sqrt{\frac{1}{6}S}$.

Thus the invariants of $I_{2,2}$ are of two types according as they correspond to invariants or covariants of f.

The best illustration of the $I_{2,2}$ is the system of intersections of a rational plane quartic curve by lines of the plane, and we shall look particularly for the invariants which correspond to obvious peculiarities of the quartic curve. Since the quartic has three double points, the $I_{2,2}$ has three neutral pairs, and on N there are three pairs of points whose polar line is indeterminate; *i. e.*, three pairs of corresponding points on Δ.

(44) *The two nets of antorthic conics can be defined as the conics which cut the Hessian of* f *in three pairs of corresponding points.*

A direct proof of this is immediate. If $A_PA_x^2$ is any polar conic of A, it satisfies the relation $A_P(aa'A)^2a_xa'_x = \sqrt{\frac{1}{6}S}\,A_PA_x^2$. But if x is on Δ, its corresponding point y is given by $(aa'u)^2a_xa'_x = \rho u_y^2$. Thus if x on Δ satisfies $A_PA_x^2 = 0$, y does the same.

That these nets contain all such conics is seen from the elliptic parametric representation of Δ. Let corresponding points u and u' satisfy the relation $u - u' \equiv \frac{\omega}{2}$ where ω and $\bar{\omega}$ are the independent periods. Then u, v, w, and their corresponding points, lie on a conic if either

$$\frac{\omega}{4} + u + v + w \equiv \begin{cases} 0 \\ \omega/2 \end{cases}$$

or

$$\frac{\omega}{4} + u + v + w \equiv \begin{cases} \bar{\omega}/2 \\ (\bar{\omega} + \omega)/2. \end{cases}$$

In the first case u and v determine a unique pair w, w' and the conics lie in a net. In the second case a similar net is obtained which is distinct from the first. These two nets must be $A_P A_x^2$ and $B_P B_x^2$.

Through u and u' there passes a pencil of each net; hence the line $\overline{uu'}$ is part of a degenerate conic in each net as well as in the net of polar conics of f. We obtain then the theorem of Professor White:

(45) *The relation of the cubics f, A, and B is mutual. They are three cubics with a common Cayleyan.*

Moreover, referring to (44):

(46) *The polar conics of A, in particular N, cut Δ and*

$$\Delta_B = -\tfrac{1}{3}(T - \sqrt{\tfrac{1}{6}S^3})(\Delta + 3\sqrt{\tfrac{1}{6}S}f),$$

the Hessian of B, each in three pairs of corresponding points.

Continuing further the parallel between the ternary and binary forms, we have

$$
\begin{aligned}
(47)\qquad s = u_s^3 \; &= (aa'a'')(aa'u)(aa''u)(a'a''u)\\
&= (a_1a_1')(a_1a_1'')(a_1'a_1'')(a_2a_2')(a_3a_3')(a_2''a_3'')(a_2\bar{x}_1)(a_2'\bar{x}_1)(a_3\bar{x}_2)\\
&\qquad\qquad (a_3'\bar{x}_2)(a_2''\bar{x}_3)(a_3''\bar{x}_3)\\
&= (\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2,\\
a_s^3 = S \; &= (\sigma_1 a_1)^2(\sigma_2 a_2)^2(\sigma_3 a_3)^2,\\
\Delta_s^3 = T \; &= (\sigma_1\delta_1)^2(\sigma_2\delta_2)^2(\sigma_3\delta_3)^2,\\
u_P = N_s^2 u_s &= (\sigma_1\sigma_2)^2(\sigma_3\bar{x})^2.
\end{aligned}
$$

By comparing the value of S just given with the value of $\sqrt{\tfrac{1}{6}S}$ in (41), a symbolic identity is obtained.

From the ternary identity

$$A_P A_s^2 u_s = \tfrac{1}{3}(T + \sqrt{\tfrac{1}{6}S^3})u_P$$

we see that *the point P in* (47) *is the pole of the norm-conic as to A.* Forming in the binary notation the polar conic of u_P as to A in (42), we find, since the conic is N_x^2,

$$(\sigma_1\sigma_2)^2(\sigma_3\delta_3)^2(\delta_1x_1)^2(\delta_2x_2)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(\sigma_3a_3)^2(a_1x_1)^2(a_2x_2)^2 = \tfrac{1}{3}(T + \sqrt{\tfrac{1}{6}S^3})(x_1x_2)^2.$$

Expanding the left-hand member into Gordan's series and equating coefficients, we obtain two identities,

$$(48)\qquad (\sigma_1\sigma_2)^2(\sigma_3\delta_3)^2(\delta_1x)^2(\delta_2x)^2+\sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(\sigma_3a_3)^2(a_1x)^2(a_2x)^2=0,$$

$$(49)\qquad (\sigma_1\sigma_2)^2(\delta_1\delta_2)^2(\sigma_3\delta_3)+\sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(a_1a_2)^2(\sigma_3a_3)^2=T+\sqrt{\tfrac{1}{6}S^3}.$$

Let $T+S\sqrt{\frac{1}{6}S}=R_1$ and $T-S\sqrt{\frac{1}{6}S}=R_2$; *i. e.*, the discriminant of f is $R=R_1R_2=T^2-\frac{1}{6}S^3$.

If in (49) the values of T, S, and $\sqrt{\frac{1}{6}S}$ be expressed by means of (47) and (41), we obtain *a syzygy among the invariants of the sixth degree of the form* $I_{2,2}$; namely,

$$(50)\qquad \begin{aligned}&(\sigma_1\delta_1)^2(\sigma_2\delta_2)^2(\sigma_3\delta_3)^2+\sqrt{\tfrac{1}{6}S}(\sigma_1a_1)^2(\sigma_2a_2)^2(\sigma_3a_3)^2\\ &=(\sigma_1\sigma_2)^2(\delta_1\delta_2)^2(\sigma_3\delta_3)^2+\sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^1(a_1a_2)^2(\sigma_3a_3)^2.\end{aligned}$$

That (50) is not a mere identity is verified by expressing the last two terms as invariants of the cubic f and point P while the first two terms are invariants of f alone. Consider the form

$$u_s^2a_sa_x^2=(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3a_3)^2(a_1x_1)^2(a_2x_2)^2.$$

The result of operating on the right side with $(x_1x_2)^2$ [on $\bar{x}_1$ and $\bar{x}_2$] and with $(\bar{x}_1\bar{x}_2)^2$ [on x_1 and x_2] is $\frac{1}{9}(\sigma_1\sigma_2)^2(\sigma_3a_3)^2(a_1a_2)^2$. Similarly operating on the left with the norm-conic in points and with the norm-conic in lines, *i. e.*, with

$$\frac{1}{R_1}A_PA_x^2 \quad\text{and}\quad \frac{1}{R_1^2}A_PA'_P(AA'u)^2,$$

we find

$$\frac{1}{R_1^3}a_sA_PA_s^2A'_PA''_P(A'A''_u)^2=\frac{1}{9}\frac{1}{R_1}a_P^3;$$

hence

$$(51)\qquad a_P^3=R_1\,.\,(\sigma_1\sigma_2)^2(a_1a_2)^2(\sigma_3a_3)^2;$$

and similarly

$$2\sqrt{\tfrac{1}{6}S}a_P^3-\Delta_P^3=R_1\,.\,(\sigma_1\sigma_2)^2(\delta_1\delta_2)^2(\sigma_3\delta_3)^2.$$

(52) *When the cubic f is non-singular, the invariant condition on the form $I_{2,2}$ that the pole P of the norm-conic N as to A shall lie on the curve $k_1B+k_2\sqrt{\frac{1}{6}S}f$ is*

$$k_1(\sigma_1\sigma_2)^2(\delta_1\delta_2)^2(\sigma_3\delta_3)-(k_2+k_1)\sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(a_1a_2)^2(\sigma_3a_3)^2=0,$$

or

$$k_2(\sigma_1\sigma_2)^2(b_1b_2)^2(\sigma_3b_3)^2-k_2\sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(a_1a_2)^2(\sigma_3a_3)^2=0.$$

Another statement of the same condition is

(53) *The pole P of N as to A lies on the curve $k_1 B + k_2 \sqrt{\frac{1}{6}S} f$ if the polar line of N (in lines) as to the curve $k_1 B - k_2 \sqrt{\frac{1}{6}S} f$ passes through P.*

In formulae (41), (47) and (52) we have the expressions for the invariants of f and the covariants of its syzygetic pencil in terms of the invariants of the form $I_{2,2}$. Other covariants would probably be not less useful. The combinants of the syzygetic pencil ought to be particularly interesting with reference to a pair of quartic envelopes defined in the next paragraph.

The following ternary identities also will be used:

$$
\begin{aligned}
(54)\quad \Delta_A &= -\tfrac{1}{3}R_1(\Delta - 3\sqrt{\tfrac{1}{6}S}f); \quad S_A = \tfrac{2}{3}R_1^2; \quad \sqrt{\tfrac{1}{6}S_A} = \pm\tfrac{1}{3}R_1. \\
\Delta_B &= -\tfrac{1}{3}R_2(\Delta + 3\sqrt{\tfrac{1}{6}S}f); \quad S_B = \tfrac{2}{3}R_2^2; \quad \sqrt{\tfrac{1}{6}S_B} = \pm\tfrac{1}{3}R_2. \\
s_A &= \tfrac{2}{3}R_1 \,.\, s; \quad A_A = \tfrac{4}{3}R_1 \,.\, \sqrt{\tfrac{1}{6}S} \,.\, f \text{ or } -\tfrac{2}{3}R_1 \,.\, B. \\
s_B &= \tfrac{2}{3}R_2 \,.\, s; \quad B_B = -\tfrac{4}{3}R_2 \,.\, \sqrt{\tfrac{1}{6}S}f \text{ or } -\tfrac{2}{3}R_2 \,.\, A.
\end{aligned}
$$

These can be employed to obtain a convenient equation for the involution conics through given points x and y. The polar conic of A through x and y is

$$
\begin{aligned}
3(AA'A'')A_x^2 A_y'^2 A_z''^2 &= 2(xyz) \,.\, \Delta_{Ax}\Delta_{Ay}\Delta_{Az} + (s_A xy)(s_A xz)(s_A yz)^* \\
&= -\tfrac{2}{3}R_1[(xyz)\{\Delta_x\Delta_y\Delta_z - 3\sqrt{\tfrac{6}{1}S}a_x a_y a_z\} - (sxy)(sxz)(syz)].
\end{aligned}
$$

Since

$$(sxy)(sxz)(syz) = (aa'a'')a_x^2 a_y'^2 a_z''^2 + 4(aa'a'')a_x a_y a_x' a_z' a_y'' a_z''$$

and

$$3(aa'a'')a_x^2 a_y'^2 a_z''^2 = 2(xyz)\Delta_x\Delta_y\Delta_z + (sxy)(sxz)(syz),$$

we find

$$(AA'A'')A_x^2 A_y'^2 A_z''^2 = \tfrac{2}{3}R_1[2(aa'a'')a_x a_y a_x' a_z' a_y'' a_z'' + \sqrt{\tfrac{1}{6}S}a_x a_y a_z \,.\, (xyz)].$$

(55) *The involution conics through x and y are*

$$2(aa'a'')a_x a_y a_x' a_z' a_y'' a_z'' \pm \sqrt{\tfrac{1}{6}S} \,.\, (xyz) \,.\, a_x a_y a_z = 0.$$

Since N is a polar conic of A, the net of polar conics of A cut out on N an $I_{1,3}$, and the lines joining the four points of a set of $I_{1,3}$ are lines of the Cayleyan. If a line of s cuts N in the points t and τ, the parametric equation of s (see § 3) is given by the *involution form* †

$$(56) \qquad (\sigma_1 t)(\sigma_2 t)(\sigma_3 t)(\sigma_1 \tau)(\sigma_2 \tau)(\sigma_3 \tau) = 0.$$

* See Gordan, *loc. cit.*, p. 10. The minus sign in the formula given there should be changed.

† That the joins of the four points of a set of $I_{1,3}$ are lines of the Cayleyan shows that the Cayleyan is the Hessian of a cubic B to which N is apolar. From this there follows the apolarity of the nets of polar conics of A and B; see Gordan, § 10.

Finally, we may emphasize in the binary domain the translation of the mutual relation of f and B with respect to A and to their Cayleyan:

(57) *The involution form $I_{2,2}=f$ has a covariant involution form $\overline{I}_{2,2}=B$ o the third degree in the coefficients. The relation of the two is mutual, and the $\overline{I}_{2,2}$ formed for $\overline{I}_{2,2}$ is $I_{2,2}$ multiplied by the invariant $-\frac{4}{3}\sqrt{\frac{1}{6}S}R_2$. $I_{2,2}$ has also a covariant involution $I_{1,3}$ of the third degree. The $I_{1,3}$ formed for $\overline{I}_{2,2}$ is $I_{1,3}$ itself multiplied by the invariant $\frac{2}{3}R_2$.*

§ 8. *The $I_{2,2}$ and the Ternary Rational Quartic.*

The given $I_{2,2}$ is cut out on N by the polar conics of B, whence the polar lines of points on N as to B envelop a rational quartic R_4 whose lines correspond to the points of N and whose $I_{2,2}$ (the four lines of R_4 through a point) corresponds to the given $I_{2,2}$ on N. In the same way, the covariant $\overline{I}_{2,2}$ on N is cut out by the polar conics of f, and the polar lines of points on N as to f envelop a rational quartic $\overline{R}_4$ whose $I_{2,2}$ is $\overline{I}_{2,2}$. In this paragraph some phases of the mutual relation between R_4 and $\overline{R}_4$ will be discussed.

Since $B=(b_1x_1)^2(b_2x_2)^2(b_3x_3)^2$, the polar conic of the point $(q\bar{x})^2$ as to B is $(b_1q)^2(b_2x_2)^2(b_3x_3)^2$, which meets N in four points whose parameters are $(b_1q)^2(b_2x)^2(b_3x)^2=0$.

(58) *The given $I_{2,2}$ is the ∞^2 quartics, for variable q, $(b_1q)^2(b_2x)^2(b_3x)^2=0$. The covariant $\overline{I}_{2,2}$ is the ∞^2 quartics $(a_1q)^2(a_1x)^2(a_2x)^2=0$.*

When $(q\bar{x})^2$ is a point on N, say $(y\bar{x})^2$, the quartic $(a_1y)^2(a_2x)^2(a_3x)^2=0$ is found in $\overline{I}_{2,2}$. But in the given $I_{2,2}$ this is four values x, each of which, taken twice with y, belongs to $I_{2,2}$. Hence

(59) *The tangent at a point of R_4 cuts R_4 in four other points whose parameters form a set of $\overline{I}_{2,2}$, and the tangent at a point of $\overline{R}_4$ cuts $\overline{R}_4$ in four other points whose parameters form a set of $I_{2,2}$. More generally, there exist on R_4 sets of four points such that the parameters of the remaining two tangents of R_4 from each point lie in a pencil of quadratics; these sets of four points lie in the $\overline{I}_{2,2}$ of $\overline{R}_4$.*

Some special cases of the antorthic $A^4_{3,3}$ call for special treatment, in particular those which are not determined when two of their points, x and y, are given. In general, the two other points are obtained as the meets of the conic $(aa'a'')a_xa_ya'_xa'_za''_ya''_z=0$ and the line $a_xa_ya_z=0$. Since the conic is not evanescent, the construction fails if (1) the conic has the line for a factor or if (2) the

line is evanescent. Denote by $Q_{x,y}$ the quadratic involutory Cremona transformation whose corresponding points are apolar to the polar conics of x and y as to f.

In case (1), the polar lines of z, any point on the polar line of xy, as to the polar conics of x and y, must meet in a point t on this polar line. Hence this polar line corresponds to itself in $Q_{x,y}$ and must be a diagonal line of the two polar conics, *i. e.*, a line of the Cayleyan. The points z and t are harmonic with the vertices on the diagonal line which are poles of the line $\overline{xy}$. From (38), if through x and y there passes a definite polar conic of A, then through x and y there must pass a pencil of polar conics of B which meet A in pairs of point z, t. Hence the polar conic of A must be the lines $(xyz)=0$ and $a_x a_y a_z = 0$. Or if u_A and v_A are a degenerate polar conic of A, each line has two of its poles as to f on the other line; the poles determine on each line a pencil of quadratics; and any member of the one pencil with any member of the other is an $A^4_{3,3}$. A similar statement holds for the degenerate polar conics of B. There are ∞^3 such $A^4_{3,3}$, ∞^1 on each proper involution conic. On N they lie in such a way that any point is contained in three $A^4_{3,3}$'s. If, in the $I_{1,3}$ of (56), t is given, three values of τ (τ_1, τ_2 and τ_3) are determined. The lines $t\tau_1$ and $\tau_2\tau_3$ form a polar conic of A, while $t\tau_1$ and the line paired with it as a polar conic of B fix an $A^4_{3,3}$ which contains t.

In case (2), the pair x, y whose $A^4_{3,3}$ is indeterminate is a pair of corresponding points on Δ. Through x and y there passes a pencil of polar conics of A and a pencil of polar conics of B. If N be taken as a fixed polar conic of A, the pencil of polar conics of B cuts out on N a quadratic involution, $I_{1,1}$.

If x and y lie on u and $Q_{x,y}$ is the Cremona involution defined earlier, then every pair of corresponding points on Δ is a pair of corresponding points in $Q_{x,y}$. Δ itself passes through the singular points of $Q_{x,y}$ which correspond on Δ to the three meets of Δ with u. The fixed points of $Q_{x,y}$ are the four poles of u. On any line v there is a pair of corresponding points in $Q_{x,y}$, say z, t, a pair whose polar line is u. On a conic there are in general four pairs of corresponding points in $Q_{x,y}$.

On the conic N let the three corresponding pairs of Δ be h_i, h'_i; let them be joined by lines H_i respectively; and let the vertex of the triangle opposite H_i be J_i; $i=1, 2, 3$. Similarly let the three corresponding pairs, k_i, k'_i, of Δ_B on N be joined by lines K_i which meet in points L_i.

If u meets N in points x, y, the four pairs of corresponding points on N in

$Q_{x,y}$ are the three pairs h_i, h_i' and the pair z, t cut out by the polar line of xy. Thus the relation of the lines $\overline{xy}$ and $\overline{zt}$ is reciprocal; they are corresponding lines in an involutory Cremona quadratic line-transformation, $\bar{Q}$, whose singular lines are H_1, H_2, H_3. Since x, y, z, t together form an $A^4_{3,3}$, we see, on allowing x and y to coincide, that

(60) *$\bar{Q}$ transforms the tangents of N into the tangents of $\overline{R}_4$, the tangent at x on N being transformed into the polar line of x as to f.*

Since lines through J_i are transformed by $\bar{Q}$ into H_i, we find that

(61) *The neutral set, h_i, h_i', forms with the point-pairs cut out on N by the pencil of lines through J_i sets of the $I^4_{2,2}$ on N.*

The $Q_{h_i, h_i'}$ has for corresponding pairs any pair of points whose polar line is $\overline{h_i h_i'}$. Since there is on N ∞^1 such pairs, N is unaltered by $Q_{h_i, h_i'}$. Evidently N goes through the singular points h_i and h_i', but it must also pass through two fixed points, say a and b; *i. e.*, two points in which the degenerate polar conics of h_i and h_i' meet. The tangents to N at a and b meet in a point from which a line pencil cuts out the corresponding points on N. According to (61) this point must be J_i, and we have the result:

(62) *If the points of contact of tangents from J_i to N are a and b, the polar conics of h_i and h_i' are respectively $\overline{h_i'a}\ \overline{h_i'b}$ and $\overline{h_i a}\ \overline{h_i b}$.*

To the three double tangents of R_4 correspond the parameters on N of the point pairs h_i, h_i'; and to the three double tangents of $\overline{R}_4$ correspond the parameters on N of the point-pairs k_i, k_i'. Since the tangents to N at a and b pass through J_i, a singular point of $\bar{Q}$, both correspond in $\bar{Q}$ to the singular line H_i. Therefore H_i is a double tangent of $\overline{R}_4$ with the parameters of a and b; *i. e.*, a and b are a pair k_i, k_i' on N. Since $\overline{ab}$ or $\overline{k_i k_i'}$ or K_i is the polar line of J_i, we find that

(63) *The triangle H_i of double tangents of $\overline{R}_4$ and the triangle K_i of double tangents of R_4 are polar triangles of N. The polar conic of h_i as to f is $\overline{h_i'k_i}\ \overline{h_i'k_i'}$; of k_i as to B is $\overline{k_i'h_i}\ \overline{k_i'h_i'}$.*

The following corollary may be worth noting:

(64) *Three pairs of corresponding points determine a cubic uniquely. If the three pairs lie on a conic, the polar triangle of their joins cuts out another set of three pairs. The two cubics constructed from the two sets of three pairs lie in a syzygetic pencil.*

The relation of the syzygetic pencil to the conic has been given above.

Another consequence of (63) is that the Jacobians of pairs of the three quadratics (on N) $h_i h_i'$ are $k_i k_i'$, and *vice versa*; and that the products of corresponding quadratics $h_i h_i' k_i k_i'$ are sets of the common covariant $I_{1,3}$ of $I_{2,2}$ and $\overline{I}_{2,2}$.

In order to define the conic N with reference to a given R_4, we recall that the tangents of R_4 from a point y are the polar lines as to B of the four points on N cut out by the polar conic of y as to B. If two of these four points coincide, y is a point of R. But the conic polar of k_1 as to B meets N at k_1' twice, and at h_1 and h_1'. Thus k_1 is a point on the double tangent K_1 where it meets R_4 again.

(65) *The conic N passes through the six points k_i, k_i' where the double tangents of R_4 meet R_4 again; and through the six points h_i, h_i' where the double tangents of $\overline{R}_4$ meet $\overline{R}_4$ again.*

Other facts as to the mutual relation of R_4 and $\overline{R}_4$ with regard to N are readily shown. For example, the given $\overline{I}_{2,2}$ has four double pairs s_i, s_i', $i=1, 2, 3, 4$. The four lines $\overline{s_i s_i'}$ are the fixed lines of $\overline{Q}$. The tangent of N at s_i is transformed by $\overline{Q}$ into the tangent at s_i' whence *the eight common tangents of $\overline{R}_4$ and N have, for parameters on N, the parameters (on R_4) of the eight tangents at the four double points of R_4*, etc.

But the statements (63) and (65) identify the curves R_4 and $\overline{R}_4$ with the pair of curves R and P which are treated at great length by Meyer,* though in dual form and in a quite different setting. Possibly the most interesting result obtained is the reciprocity between three pairs of corresponding points of Δ on a conic and the six further intersections of the polar conics of the six points with the conic.

§ 9. *Interpretation of Certain Invariants of the Form $I_{2,2}$.*

The rational quartic loci just obtained have been of class four. The curve of order four, $R^{(4)}$, is handled more commonly, and results will be stated for this curve even though they have been gotten in the dual form.

From the obvious properties of the $I_{2,2}$ there follows:

(66) *The curves f and Δ cut out on N the points whose parameters on N are the parameters on $R^{(4)}$ of the six flexes and the three double points respectively; i. e., the six flexes and the three double points are determined by the equations:*

$$(a_1x)^2(a_2x)^2(a_3x)^2=0$$

and

$$(\delta_1x)^2(\delta_2x)^2(\delta_3x)^2=0$$

respectively.

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* *Apolarität*, pp. 258–272.

The three cubics f, A, and B lie in the syzygetic pencil of f and Δ. The polar conics of P as to three cubics lie in a pencil whose base-points are an $A^4_{3,3}$ of each curve of the syzygetic pencil. To the ∞^2 positions of the pole P correspond ∞^2 such antorthic sets of the syzygetic pencil,* *one on each involution conic.* On N this set of four points is the common tetrad of $I_{2,2}$ and $\bar{I}_{2,2}$. Since it is the four points in which the polar conic of P (the pole of N as to A) as to f meets N, its equation is (see 47)

$$(67) \qquad (\sigma_1\sigma_2)^2(\sigma_3 a_3)^2(a_1 x)^2(a_2 x)^2 = 0.$$

According to Caporali,† if three points of $A^4_{3,3}$ lie on f, the fourth also lies on f, and the four points are the meets, other than P, of f and the polar conic as to f of a point P on f. Hence

(68) *If three flexes of the curve $R^{(4)}$ lie on a line, a fourth also lies on the line.*‡ *The invariant condition that four flexes lie on a line is that the pole P lie on f or, in terms of the coefficients of $I_{2,2}$, that the invariant of degree four* (see 52)

$$(\sigma_1\sigma_2)^2(a_1 a_2)^2(\sigma_3 a_3)^2$$

vanish. If this condition is satisfied, the four flexes are obtained from the equation (67).

In considering invariants of $I_{2,2}$ which are at the same time invariants of f, one must examine each particular case with regard to possible variations in the number of involution conics and with regard to the determination of the pole P. In this connection the list of canonical forms and their covariants tabulated by Gordan § for all special cubics f is very convenient.

$S = 0$ is the condition that f have an antorthic set of three points such that any two of the three are apolar to f; *i. e.*, an $A^3_{2,3}$. Then the ∞^4 $A^4_{3,3}$'s of f are distributed, ∞^2 at a time, on the conics of the net through $A^3_{2,3}$. Let the parameters of $A^3_{2,3}$ on N, any one of the involution conics, be μ_1, μ_2, μ_3. The polar conic of the point μ_i as to f is the square of the line $\overline{\mu_k\mu_l}$. Thus, for the given value μ_i, $I_{2,2}$ reduces to an $I_{1,2}$ which consists of the neutral pair $\mu_k\mu_l$ and an arbitrary point (§6). Hence $I_{2,2}$ contains the neutral triad $\mu_1\mu_2\mu_3$ and $R^{(4)}$ has a triple point.

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* Caporali, *loc. cit.*, p. 52, § 23.

† *Loc. cit.*, p. 52, § 22.

‡ Brill, *Math. Annalen*, Vol. XII.

§ *Trans. American Math. Soc.*, Vol. I, p. 492.

(69) *The vanishing of the invariant* $\sqrt{\frac{1}{6}S}$ *of the second degree in the coefficients of* $I_{2,2}$ *is the condition that* $R^{(4)}$ *have a triple point.*

To express certain conditions on f, the contravariant t, the evectant of the invariant T, is necessary. Let

(70) $$t = u_t^3 = (t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2.$$

Since $\sqrt{\frac{1}{6}S}$ is rational in the coefficients of $I_{2,2}$, the discriminant of f, $R = T^2 - \frac{1}{6}S^3$, breaks up into two rational factors, $R_1 = T + S\sqrt{\frac{1}{6}S}$ and $R_2 = T - S\sqrt{\frac{1}{6}S}$. To determine the involution conics, we take f in the canonical form

$$\begin{aligned}
f &= \quad x_1^3 + x_2^3 + 6x_1x_2x_3,\\
\Delta &= -6(x_1^3 + x_2^3 - 2x_1x_2x_3),\\
S &= \quad 24, \quad T = 48, \quad \pm\sqrt{\tfrac{1}{6}S} = \pm 2,\\
\Delta - 2f &= -8(x_1^3 + x_2^3),\\
\Delta + 2f &= -4(x_1^3 + x_2^3 - 6x_1x_2x_3).
\end{aligned}$$

From the formula (55), or directly from the relation (36) for $\lambda = \pm 2$, two nets of involution conics are found:

$$\rho_1x_1^2 + \rho_2x_2^2 + \rho_3(2x_3^2 - x_1x_2)$$
$$\sigma_1(x_1^2 - 2x_2x_3) + \sigma_2(x_2^2 - 2x_3x_1) + \sigma_3x_1x_2.$$

The conics in the second net are the polar conics of $\Delta + 2f$, each conic having a definite pole. The conics of the first net are not all polar conics, and in this case the correspondence between pole and conic is no longer unique. But from (47′) the correspondence between pole and conic fails when $R_1 = 0$. Hence $R_1 = 0$ is the condition that N be a conic of the first net, and $R_2 = 0$ is the condition that N be a conic of the second net. From the usual* parametric representation of Δ, it is easily verified that the conics of the first net cut Δ in three pairs of corresponding points, while those of the second net cut Δ in two pairs of corresponding points in addition to the self-corresponding point, the common double point of f and Δ.

* Cf. Clebsch–Lindemann, Vol. II, p. 336. In the same chapter the invariant conditions on f to be used later are found.

Let $R_2 = 0$, whence N is a conic of the second net on which the double point of f has a parameter μ_1. Since the polar conic of μ_1 as to f is a pair of lines which meet at μ_1, for the value μ_1, $I_{2,2}$ reduces to an $I_{1,2}$ which has a neutral point μ_1, (§6). Hence $I_{2,2}$ arises from a curve $R^{(4)}$ with a cusp.

(71) *The vanishing of the invariant, $T - S\sqrt{\frac{1}{6}S}$, of the sixth degree in the coefficients of $I_{2,2}$ is the condition that $R^{(4)}$ have a cusp.*

When f has a double point, the contravariant $\Pi = St - Ts$ is the cube of the double point. Since now $T = S\sqrt{\frac{1}{6}S}$, this is $S(t - \sqrt{\frac{1}{6}S}s)$, or, in the binary notation, Π is proportional to

$$(t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2 - \sqrt{\tfrac{1}{6}S}(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2.$$

Hence

(72) *When $T - S\sqrt{\frac{1}{6}S} = 0$, the parameter of the cusp is determined from the equation $(t_1x)^2(t_2x)^2(t_3x)^2 - \sqrt{\frac{1}{6}S}(\sigma_1x)^2(\sigma_2x)^2(\sigma_3x)^2 = 0$, which is a perfect sixth power.*

Let $R_1 = 0$, whence N is a conic of the first net. Evidently the cubics f and $B = \Delta + 2f$ are interchanged by the harmonic perspectivity with the double point of f, $\xi_3 = 0$, as center and the line of flexes of f, $x_3 = 0$, as axis. But this perspectivity leaves N unaltered. Hence the envelopes of class four, R_4 and $\bar{R}_4$, of the preceding paragraph are interchanged by the perspectivity, as also are their triangles of double tangents H_i and $\bar{K}_i$. On N the perspectivity is an ordinary quadratic involution which interchanges the pairs of points h_i, h_i' with k_i, k_i'. Since Π is the cube of ξ_3 and ξ_3 does not lie on N (when N is a proper conic), the polar of N as to Π is ξ_3. Π is now proportional to

$$(t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2,$$

and the polar of N as to Π is represented by the quadratic $(t_1t_2)^2(t_3\bar{x}_3)^2 + \sqrt{\frac{1}{6}S}(\sigma_1\sigma_2)^2(\sigma_3\bar{x})^2$. But this quadratic determines the fixed points of the perspectivity on N.

(73) *The vanishing of the invariant, $T + S\sqrt{\frac{1}{6}S}$, of the sixth degree in the coefficients of $I_{2,2}$ is the condition that there shall exist a binary involution on $R^{(4)}$ which interchanges the parameters of each double point of $R^{(4)}$ with the parameters of*

the two tangents from the double point. The covariant $(t_1t_2)^2(t_3x)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\sigma_2)^2(\sigma_3x)^2$ *of the sixth degree determines the fixed points of the involution.**

Let $S=0$ and $T=0$. Then f has a cusp, and there is a single net of involution conics any one of which, say N, passes through the cusp with a parameter μ_1, through the meet of the cusp tangent and inflexion tangent with a parameter μ_2, and touches at μ_1 the join of the inflexion and cusp. Since the polar conic of μ_1 as to f is the square of the line $\overline{\mu_1\mu_2}$, $I_{2,2}$ has ∞^1 tetrads consisting of μ_1 twice, μ_2, and an arbitrary point.

(74) *The vanishing of the invariants,* $\sqrt{\tfrac{1}{6}S}$ *and* T, *of degree two and six is the condition that* $R^{(4)}$ *have a triple point with two coincident branches.*

Let f have two double points. Then

$$f = x_1^3 + 6x_1x_2x_3, \qquad \Delta = -6(x_1^3 - 2x_1x_2x_4),$$
$$S = 24, \qquad T = 48, \qquad \pm\sqrt{\tfrac{1}{6}S} = \pm 2,$$
$$\Delta - 2f = -8x_1^3, \qquad \Delta + 2f = -4(x_1^3 - 6x_1x_2x_3).$$

Corresponding to the values $\lambda = -2$ and $\lambda = +2$, there are respectively the two nets of involution conics

$$\rho_1x_1^2 + \rho_2x_2^2 + \rho_3x_3^2,$$
$$\sigma_1(x_1^2 - 2x_2x_3) + \sigma_2x_3x_1 + \sigma_3x_1x_2.$$

As before, the two nets are distinguished by the vanishing of R_1 and R_2 respectively. When $R_2=0$, the conic N passes through the two double points and the curve $R^{(4)}$ has two cusps. The two double points are cut out on N by $\Delta - 2f$ three times. The contravariant Π vanishes identically. Hence

(75) *When* $T - S\sqrt{\tfrac{1}{6}S} = 0$ *and*

$$(t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2 - \sqrt{\tfrac{1}{6}S}(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2 \equiv 0,$$

the curve $R^{(4)}$ *has two cusps which are determined by the equation with two triple roots*

$$(\delta_1x)^2(\delta_2x)^2(\delta_3x)^2 + \sqrt{\tfrac{1}{6}S}(a_1x)^2(a_2x)^2(a_3x)^2 = 0.$$

* We may note here that the theorem of Meyer (*Apolarität*, p. 260, footnote): "Es giebt eine bestimmte projectivische Beziehung, die irgend drei Elementenpaare in ihre bezüglichen Funktionaldeterminanten überführt" is not correct. Another version of the statement is:—Two perspective triangles (*i. e.*, two polar triangles as to a conic) can be interchanged by a collineation which is necessarily involutory;—and this is true only when corresponding vertices of the triangles are harmonically separated by the center and axis of perspection. This is one condition on the two perspective triangles or one condition on one triangle and the conic. Hence the three quadratics in the theorem quoted must be subject to one condition.

The error is the result of another in the statement of the theorem, (π), p. 260. Only when the invariant $T + S\sqrt{\tfrac{1}{6}S}$ of the $I_{2,2}$ of either curve R and P vanishes is the theorem (π) correct.

On the other hand, let $R_1 = 0$, whence N is a conic of the first net unaltered by the three harmonic perspectivities whose centers and axes are (ξ_1, x_1), (ξ_2, x_2), and (ξ_3, x_3). The last two interchange f and B, while the first leaves each unaltered. The two envelopes, R_4 and $\bar{R}_4$, are again mutually related, each being unaltered by one perspectivity and the two being interchanged by the other two perspectivities. N cuts the common line, $x_1 = 0$, of f and Δ in two points with parameters μ_1 and μ_2 and cuts Δ further in two pairs of points, each pair being harmonically separated by $x_1 = 0$ and $\xi_1 = 0$. Hence the perspectivity (ξ_1, x_1) leaves the three double tangents of R_4 (as well as of $\bar{R}_4$) each unaltered, the points of contact on one double tangent being unaltered, while those on the other two are interchanged. Since $\Delta - 2f = -8x_1^3$, the polar of N in lines as to $\Delta - 2f$ is the line $x_1 = 0$. Hence

(76) *When* $T + S\sqrt{{}^1S} = 0$ *and*

$$(t_1\bar{x}_1)^2(t_2\bar{x}_2)^2(t_3\bar{x}_3)^2 + \sqrt{\tfrac{1}{6}S}(\sigma_1\bar{x}_1)^2(\sigma_1\bar{x}_2)^2(\sigma_3\bar{x}_3)^2 \equiv 0,$$

the curve $R^{(4)}$ is unaltered by a harmonic perspectivity with center at one double point and axis through the other two. The first double point is also a double flex-point. The parameters of the double flex-point are determined by the equation

$$(\delta_1\delta_2)^2(\delta_3x)^2 + \sqrt{\tfrac{1}{6}S}(a_1a_2)^2(a_3x)^2 = 0.$$

There exist on the curve two binary involutions which interchange the parameters at the double flex-point, and, at each of the other double points, interchange the parameters of the double point with the parameters of tangents from the double point.

Let f have three double points. Then

$$f = 6x_1x_2x_3, \qquad \Delta = 12x_1x_2x_3, \qquad S = 24, \qquad T = 48,$$
$$\pm\sqrt{\tfrac{1}{6}S} = \pm 2, \qquad \Delta - 2f \equiv 0, \qquad \Delta + 2f = 24x_1x_2x_3.$$

The involution conics are again in two nets,

$$\rho_1x_1^2 + \rho_2x_2^2 + \rho_3x_3^2,$$
$$\sigma_1x_2x_3 + \sigma_2x_3x_1 + \sigma_3x_1x_2,$$

corresponding respectively to $R_1 = 0$ and $R_2 = 0$. For conics of the second net $R^{(4)}$ has three cusps; for those of the first net, since f and Δ are proportional, the double points coincide with the flexes. The invariant condition in the ternary domain is $\Delta - 2f \equiv 0$.

(77) *When* $T - S\sqrt{\frac{1}{6}S} = 0$ *and*

$$(\delta_1 x_1)^2(\delta_2 x_2)^2(\delta_3 x_3)^2 + \sqrt{\tfrac{1}{6}S}(a_1 x_1)^2(a_2 x_2)^2(a_3 x_3)^2 \equiv 0,$$

the curve $R^{(4)}$ *has three cusps; when, however,* $T + S\sqrt{\frac{1}{6}S} = 0$ *and the same covariant vanishes, the curve* $R^{(4)}$ *has three double flex-points.*

Let f be a conic and one of its tangents.

$$f = 3x_1(x_2^2 + x_3 x_1), \qquad \Delta = -6x_1^3, \qquad S = T = 0, \qquad t \equiv 0.$$

The involution conics are now in a single net,

$$\rho_1 x_1^2 + \rho_2 x_1 x_2 + \rho_3(x_2^2 - x_3 x_1),$$

whose members touch the line of f, $x_1 = 0$, at the point, $\xi_3 = 0$, where it touches the conic of f. Hence N meets Δ at this point only with a parameter μ_1. The polar conic of μ_1 as to f is the square of a line which touches N at μ_1. For the value μ_1, $I_{2,2}$ reduces to an $I_{1,2}$ consisting of μ_1 twice and an arbitrary point. Thus the curve $R^{(4)}$ has a triple point which is formed by the coalescence of two cusps and a node into a smooth point. Such a singularity, the reciprocal of an undulation point, will be called a bi-stationary point.

(78) *The identical vanishing of* $(t_1 x_1)^2(t_2 x_2)^2(t_3 x_3)^2$ *is the condition that* $R^{(4)}$ *have a bi-stationary point. The parameter of the point is determined as the six-fold root of the equation* $(\delta_1 x)^2(\delta_2 x)^2(\delta_3 x)^2 = 0$.

For further degenerations of the cubic f into three lines the involution conics are also all degenerate and the above enumeration of types of $R^{(4)}$ which correspond to types of the cubic f is complete. But for every type of f there will exist sub-types of $R^{(4)}$ which correspond to special positions of the pole P of N. Without attempting a complete discussion of these cases, some of especial interest may be pointed out.

If the polar conic of P as to A touches Δ at one point, it must touch Δ at the corresponding point also, and on $R^{(4)}$ two double points come together to form a tac-node. Since the polar conics of A which touch Δ form a singly infinite system, their poles lie on a locus Γ which we shall determine. Γ is also the envelope of the polar lines as to A of points on Δ. Such an envelope is in general of the sixth class, but since the polar lines of corresponding points on Δ are the same, the class of Γ is three. Δ and A are each unaltered by the collineation G_{18} of the syzygetic pencil; Γ is then unaltered by the same group and can be assumed as

$$u_1^3 + u_2^3 + u_3^3 + 6\gamma u_1 u_2 u_3.$$

Here γ is determined by requiring the locus Γ to touch the polar line as to $A = \Sigma x_1^3 + 6ax_1x_2x_3$ of the point $0, 1, -1$, a flex-point on Δ. This polar line is $-2ax_1 + x_2 + x_3 = 0$, whence $-8a^3 + 2 - 12a\gamma = 0$. If $f = \Sigma x_1^3 + 6mx_1x_2x_3$, then a satisfies the relation, $4ma^2 + 4m^2a + 1 = 0$. Lowering, by means of this relation, powers of a higher than the first, the condition on γ becomes $-4m^3 + 1 - 6m\gamma = 0$, whence $\gamma = \dfrac{1 - 4m^3}{6m}$ and Γ is the common Cayleyan of f, A, and B.

(79) *If s is the common Cayleyan of three cubics, the polar conic of a point P on s as to one of the cubics touches each of the Hessians of the other two cubics at a pair of corresponding points.*

If the polar conic of P as to A passes through a common point of f and Δ, *i. e*, if P lies on a flex-tangent of A, then $R^{(4)}$ has a flex-point at a double point. Hence

(80) *The invariant condition on the coefficients of $I_{2,2}$ in order that $R^{(4)}$ have a tac-node is that the pole $P = (\sigma_1\sigma_2)^2(\sigma_3\bar{x})^2$ shall lie on the Cayleyan $(\sigma_1\bar{x}_1)^2(\sigma_2\bar{x}_2)^2(\sigma_3\bar{x}_3)^2$; the condition that $R^{(4)}$ have a flex at a double point is that the pole P shall lie on one of the nine flex-tangents of the cubic*

$$(\delta_1x_1)^2(\delta_2x_2)^2(\delta_3x_3)^2 + \sqrt{\tfrac{1}{6}S}(a_1x_1)^2(a_2x_2)^2(a_3x_3)^2.$$

These invariant conditions are not given in symbolic form, because a direct substitution of the coordinates of P in the locus in question gives the required invariant multiplied by extraneous invariants.* Until more is known of the relations among the invariants of the form $I_{2,2}$, explicit expressions are out of the question. Take, for instance, the condition that $R^{(4)}$ have an undulation point, two coincident flex-points. Then $N = (x_1x_2)^2$ must touch f. The tact-invariant of the conic and the cubic is of degree ten in the coefficients of the cubic. Since the undulation condition is known to be of degree four, an invariant of degree six must factor out of the tact-invariant. This extraneous factor must be R_2, since for $R_2 = 0$, N touches f (*i. e.*, passes through the double point of f), and on $R^{(4)}$ two flexes coincide (at the cusp of R_4).

The tac-node condition, obtained by substitution of the pole P, as in (80), or by forming the tact-invariant of N and Δ, appears in either case as an invariant of degree thirty. Since two contacts with Δ are involved simultaneously,

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* *E. g.*, the condition that P lie on a curve of the syzygetic pencil is of degree twelve when obtained by substitution, but contains the factor R_1. See (52).

this is probably a perfect square from which R_1 also factors, leaving an invariant of degree nine as the probable true tac-node condition.

The true undulation-condition of degree four in the coefficients of $I_{2,2}$ can be obtained as follows: On $R^{(4)}$ the $I^4_{2,2}$ determines the apolar involution $I^4_{1,3}$, the so-called "fundamental involution on $R^{(4)}$." $I^4_{1,3}$ is a linear covariant of $I^4_{2,2}$. It is, in fact,

$$(r_1t_1)^3(r_2t_2)^3 \equiv 2(a_1t_1)(a_2t_1)(a_3t_1)(a_1t_2)(a_2t_2)(a_3t_2) - (t_1t_2)^2(a_1a_2)^2(a_3t_1)(a_3t_2)$$

if t_1 and t_2 belong to a tetrad of $I^4_{1,3}$. This form has been represented as a quadric in S_3, the vanishing of whose discriminant [given symbolically in (16)] is the condition that all the tetrads of $I^4_{1,3}$ have a common point which is an undulation point on $R^{(4)}$.

Finally, it may be worth while to give the point-equations of the class quartics, R_4 and R_4. Since R_4 is the envelope of the polar lines of points on N as to B, it is also the locus of points, y, whose polar conics as to B touch N. The invariants of the two conics, $N = A_PA^2_x$ and $B_yB^2_x$, are:

$$\begin{aligned}(81)\qquad A_{111} &= A_PA'_PA''_P(AA'A'')^2 = -\tfrac{1}{3}R_1[\Delta^3_P - 3\sqrt{\tfrac{1}{6}S}f^3_P],\\ A_{112} &= A_PA'_PB_y(AA'B)^2 \;= -\tfrac{1}{3}R_1[B^2_PB_y],\\ A_{122} &= A_PB_yB'_y(ABB')^2 \;= -\tfrac{1}{3}R_2[A_PA^2_y],\\ A_{222} &= B_yB'_yB''_y(BB'B'')^2 = -\tfrac{1}{3}R_2[\Delta^3_y - 3\sqrt{\tfrac{1}{6}S}f^3_y].\end{aligned}$$

To obtain $\overline{R}_4$, f replaces B and the conic $a_ya^2_x$ replaces $B_yB^2_x$. The invariants of N and $a_ya^2_x$ are:

$$\begin{aligned}(82)\qquad \overline{A}_{111} &= A_{111} = -\tfrac{1}{3}R_1[\Delta^3_P - 3\sqrt{\tfrac{1}{6}S}f^3_P],\\ \overline{A}_{112} &= \qquad\quad \tfrac{1}{3}R_1[a^2_Pa_y],\\ \overline{A}_{122} &= \qquad\quad \sqrt{\tfrac{1}{6}S}[A_PA^2_y],\\ \overline{A}_{222} &= \qquad\quad \Delta^3_y.\end{aligned}$$

In terms of these, the tact-invariant of N and $B_yB^2_x$ is*

$$4(A_{111}A_{122} - A^2_{112})(A_{112}A_{222} - A^2_{122}) - (A_{111}A_{222} - A_{112}A_{122})^2 = 0,$$

which is the sextic point-equation in variables y of the quartic R_4.

The locus of points y from which the tangents to R_4 are equianharmonic is

$$2i = 3(A^2_{112} - A_{122}A_{111}) = 0,$$

which is the conic through the cusps of R_4. Since A_{122} is the conic N, we find

* Clebsch-Lindemann, Vol. I, p. 371 et seq.

dually that the conic which touches the six flex-tangent of the point-quartic $R^{(4)}$, and the conic which touches the tangents from the three double points of $R^{(4)}$ *touch at two points.*

The locus of points y from which the tangents to R_4 are harmonic is the cubic

$$8/3j = 3A_{111}A_{112}A_{122} - A_{211}^2A_{222} - 2A_{112}^3 = 0.$$

From these formulae the locus of points y from which the tangents to R_4 have any desired anharmonic ratio (a sextic curve) can be gotten.

The point-equation of R_4 is symmetrical in y and B, and P and A. The locus of points for which R_4 passes through P is a curve of order twelve, the product of the four flex-triangles. Thus the condition that the pole P lie on a flex-triangle is the discriminant of the quartic covariant of (67) and is, at least formally, of degree twenty-four. There seems to be no simple geometrical peculiarity of $R^{(4)}$ which corresponds to this invariant.

In an article to appear in the next number of this Journal, the relation between the rational space quintic curve and the plane quartic curve (subject to one condition), and the relation between the rational plane quintic curve and the cubic surface in S_3 will be discussed.

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